

A general existence and uniqueness result on multidimensional BSDEs [☆]

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Abstract

This paper establishes a new existence and uniqueness result of solutions for multidimensional backward stochastic differential equations (BSDEs) whose generators satisfy a weak monotonicity condition and a general growth condition in y , which generalizes the corresponding results in [2], [3] and [5].

Keywords: Backward stochastic differential equation, Existence and uniqueness, Weakly monotonic condition, Lipschitz condition, Mao's condition

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1. Introduction

In this paper, we are concerned with the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T], \quad (1)$$

where $T > 0$ is a constant called the time horizon, ξ is a k -dimensional random vector called the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$ is progressively measurable for each (y, z) , called the generator of BSDE (1), and B is a d -dimensional Brownian motion. BSDE (1) is denoted by BSDE (ξ, T, g) . The solution (y, z) is a pair of adapted processes.

BSDEs were initially introduced in a nonlinear form in 1990 by Pardoux and Peng [4], who established an existence and uniqueness result for the adapted and squared integrable solutions of BSDEs under the Lipschitz assumption of the generator g . From then on, many researchers have been working on this subject, and many applications have been found in mathematical finance, stochastic control, and partial differential equations, etc. In particular,

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an interesting and important question is how to improve the existence and uniqueness result of [4] by weakening the Lipschitz continuity condition on the generator g . Here, we would like to cite some efforts devoted to this direction and related closely to this paper. In 1995, Mao [3] obtained an existence and uniqueness result of a solution for (1) where g satisfies some kind of non-Lipschitz condition in y called usually the Mao's condition. In 1999, Pardoux [5] established an existence and uniqueness result of a solution for (1) where g satisfies some kind of monotonicity condition and a general growth condition in y . Furthermore, in 2003, using the same monotonicity condition as in [5] and a more general growth condition in y for g , Briand et al. [1] investigated the existence and uniqueness of a solution for (1). Recently, under the general growth condition employed in [5] as well as a weaker monotonicity condition in y for g , Fan and Jiang [2] proved an existence and uniqueness result of a solution for (1), which unifies the results obtained in [3] and [5].

The objective of this paper is to further generalize the existence and uniqueness result obtained in [2]. We establish a new existence and uniqueness result for solutions of multidimensional BSDEs whose generators satisfy the weaker monotonicity condition in y put forward by [2] and the more general growth condition in y employed in [1] (see Theorem 1 in Section 3), which generalizes the corresponding results in [3], [5] and [2]. Particularly, it should be mentioned that the integrability condition on the process $\{g(t, 0, 0)\}_{t \in [0, T]}$ used in [2] is also weakened in Theorem 1 of this paper. The remainder is organized as follows. We introduce some preliminaries and establish a technical proposition in Section 2, and put forward and prove our main result in Section 3.

2. Preliminaries

Let us fix a number $T > 0$, and two positive integers k and d . Let (Ω, \mathcal{F}, P) be a probability space carrying a standard d -dimensional Brownian motion $(B_t)_{t \geq 0}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural σ -algebra generated by $(B_t)_{t \geq 0}$ and $\mathcal{F} = \mathcal{F}_T$. In this paper, the Euclidean norm of a vector $y \in \mathbb{R}^k$ will be defined by $|y|$, and for an $k \times d$ matrix z , we define $|z| = \sqrt{\text{Tr}(zz^*)}$, where z^* is the transpose of z . Let $\langle x, y \rangle$ represent the inner product of $x, y \in \mathbb{R}^k$. We denote by $L^2(\mathcal{F}_T; \mathbb{R}^k)$ the set of all \mathbb{R}^k -valued, square integral and \mathcal{F}_T -measurable random vectors. Let $\mathcal{S}^2(0, T; \mathbb{R}^k)$ denote the set of \mathbb{R}^k -valued, adapted and continuous processes $(\phi_t)_{t \in [0, T]}$ such that

$$\|\phi\|_{\mathcal{S}^2}^2 := \mathbb{E} \left[\sup_{t \in [0, T]} |\phi_t|^2 \right] < +\infty.$$

Moreover, let $M^2(0, T; \mathbb{R}^{k \times d})$ denote the set of (\mathcal{F}_t) -progressively measurable $\mathbb{R}^{k \times d}$ -valued processes $(\varphi_t)_{t \in [0, T]}$ such that

$$\|\varphi\|_{M^2}^2 := \mathbb{E} \left[\int_0^T |\varphi_t|^2 dt \right] < +\infty.$$

Obviously, $\mathcal{S}^2(0, T; \mathbb{R}^k)$ is a Banach space and $M^2(0, T; \mathbb{R}^{k \times d})$ is a Hilbert space.

As mentioned in the introduction, we will deal only with BSDEs which are equations of type (1), where the terminal condition $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^k)$, and the generator g is (\mathcal{F}_t) -progressively measurable for each (y, z) .

Definition 1 A pair of processes $(y_t, z_t)_{t \in [0, T]}$ is called a solution to BSDE (1), if $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ and satisfies (1).

Now, let us introduce the following Proposition 1, which will play an important role in the proof of our main result. In stating it, the following assumption on the generator g is useful:

$$(A) \quad dP \times dt - a.e., \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}, \quad \langle y, g(\omega, t, y, z) \rangle \leq \psi(|y|^2) + \lambda|y||z| + |y|f_t,$$

where $\lambda > 0$ is a constant, $(f_t)_{t \in [0, T]}$ is a nonnegative and (\mathcal{F}_t) -measurable process with

$$\mathbb{E} \left[\left(\int_0^T f_t dt \right)^2 \right] < +\infty,$$

and $\psi(\cdot)$ is a nondecreasing and concave function from \mathbb{R}^+ to itself with $\psi(0) = 0$.

Proposition 1 Let g satisfy (A) and $(y_t, z_t)_{t \in [0, T]}$ be a solution to BSDE (ξ, T, g) . Then there exists a constant $C > 0$ depending only on λ and T such that for each $0 \leq u \leq t \leq T$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + \mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \\ & \leq C \left\{ \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + \int_t^T \psi(\mathbb{E} [|y_s|^2 | \mathcal{F}_u]) ds + \mathbb{E} \left[\left(\int_t^T f_s ds \right)^2 \middle| \mathcal{F}_u \right] \right\}. \end{aligned}$$

Proof. Applying Itô's formula to $|y_t|^2$ leads that for each $t \in [0, T]$,

$$|y_t|^2 + \int_t^T |z_s|^2 ds = |\xi|^2 + 2 \int_t^T \langle y_s, g(s, y_s, z_s) \rangle ds - 2 \int_t^T \langle y_s, z_s dB_s \rangle. \quad (2)$$

By assumption (A) and the inequality $2ab \leq 2a^2 + b^2/2$ we have

$$\begin{aligned} 2\langle y_s, g(s, y_s, z_s) \rangle & \leq 2\psi(|y_s|^2) + 2\lambda|y_s||z_s| + 2|y_s|f_s \\ & \leq 2\psi(|y_s|^2) + 2\lambda^2|y_s|^2 + \frac{1}{2}|z_s|^2 + 2|y_s|f_s. \end{aligned} \quad (3)$$

It follows from the Burkholder-Davis-Gundy inequality that $\{M_t := \int_0^t \langle y_s, z_s dB_s \rangle\}_{t \in [0, T]}$ is a uniformly integrable martingale. In fact, for each $0 \leq u \leq t \leq T$, we have

$$\begin{aligned} 2\mathbb{E} \left[\sup_{r \in [t, T]} \left| \int_r^T \langle y_s, z_s dB_s \rangle \right| \middle| \mathcal{F}_u \right] & \leq 2c\mathbb{E} \left[\sup_{r \in [t, T]} |y_r| \cdot \left(\int_t^T |z_s|^2 ds \right)^{1/2} \middle| \mathcal{F}_u \right] \\ & \leq \frac{1}{2}\mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + 2c^2\mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \\ & < +\infty, \end{aligned} \quad (4)$$

where $c > 0$ is a constant. Then, it follows from (2), (3) and (4) that for each $0 \leq u \leq t \leq T$,

$$\frac{1}{2}\mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \leq \mathbb{E} [X_t | \mathcal{F}_u] + 2\mathbb{E} \left[\int_t^T |y_s|f_s ds \middle| \mathcal{F}_u \right], \quad (5)$$

where

$$X_t = |\xi|^2 + 2\lambda^2 \int_t^T |y_s|^2 ds + 2 \int_t^T \psi(|y_s|^2) ds.$$

Furthermore, by virtue of (3), (4) and the following inequality

$$\begin{aligned} 2\mathbb{E} \left[\int_t^T |y_s| f_s ds \middle| \mathcal{F}_u \right] &\leq 2\mathbb{E} \left[\sup_{r \in [t, T]} |y_r| \cdot \int_t^T f_s ds \middle| \mathcal{F}_u \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + 4\mathbb{E} \left[\left(\int_t^T f_s ds \right)^2 \middle| \mathcal{F}_u \right], \end{aligned} \quad (6)$$

it follows from (2) that for each $0 \leq u \leq t \leq T$,

$$\begin{aligned} &\frac{1}{4} \mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \\ &\leq \mathbb{E} [X_t | \mathcal{F}_u] + 4\mathbb{E} \left[\left(\int_t^T f_s ds \right)^2 \middle| \mathcal{F}_u \right] + 2c^2 \mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right]. \end{aligned}$$

Combining the above inequality, (5) and (6) with 4 being replaced by $32c^2$ yields that for each $0 \leq u \leq t \leq T$,

$$\begin{aligned} &\frac{1}{8} \mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \\ &\leq (4c^2 + 1) \mathbb{E} [X_t | \mathcal{F}_u] + (16c^2 + 4) \mathbb{E} \left[\left(\int_t^T f_s ds \right)^2 \middle| \mathcal{F}_u \right], \end{aligned}$$

and then, in view of the definition of X_t , Fubini's theorem, the concavity of $\psi(\cdot)$ and Jensen's inequality, we have

$$\begin{aligned} &\frac{1}{8} \mathbb{E} \left[\sup_{r \in [t, T]} |y_r|^2 \middle| \mathcal{F}_u \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_u \right] \\ &\leq (4c^2 + 1) \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + 2(4c^2 + 1) \int_t^T \psi(\mathbb{E} [|y_s|^2 | \mathcal{F}_u]) ds \\ &\quad + (16c^2 + 4) \mathbb{E} \left[\left(\int_t^T f_s ds \right)^2 \middle| \mathcal{F}_u \right] + 2\lambda^2 (4c^2 + 1) \int_t^T \mathbb{E} \left[\sup_{r \in [s, T]} |y_r|^2 \middle| \mathcal{F}_u \right] ds, \end{aligned}$$

from which together with Gronwall's inequality, the desired result follows. The proof is then completed.

Remark 1 Proposition 1 improves the corresponding result in [2], where the process (f_t) defined in assumption (A) is assumed to satisfy the condition that

$$\mathbb{E} \left[\int_0^T |f_t|^2 dt \right] < +\infty.$$

3. Main result and its proof

In this section, we will put forward and prove our main result. Let us first introduce the following assumptions on the generator g :

(H1) g satisfies the weakly monotonic condition in y , i.e., there exists a nondecreasing and concave function $\kappa(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\kappa(u)} = +\infty$ such that $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbb{R}^k, z \in \mathbb{R}^{k \times d}, \quad \langle y_1 - y_2, g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \rangle \leq \kappa(|y_1 - y_2|^2).$$

(H2) $dP \times dt - a.e.$, $\forall z \in \mathbb{R}^{k \times d}$, $y \mapsto g(\omega, t, y, z)$ is continuous.

(H3) $\forall \alpha > 0$, $\phi_\alpha(t) := \sup_{|y| \leq \alpha} |g(\omega, t, y, 0) - g(\omega, t, 0, 0)| \in L^1([0, T] \times \Omega)$.

(H4) g is Lipschitz continuous in z uniformly with respect to (ω, t, y) , i.e., there exists a constant $\mu \geq 0$ such that $dP \times dt - a.e.$,

$$\forall y \in \mathbb{R}^k, z_1, z_2 \in \mathbb{R}^{k \times d}, \quad |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \mu |z_1 - z_2|.$$

(H5) $\mathbb{E} \left[\left(\int_0^T |g(\omega, t, 0, 0)| dt \right)^2 \right] < +\infty$.

In this paper, we want to obtain an existence and uniqueness result for BSDE (1) under the previous assumptions (H1)-(H5) and $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^k)$. Firstly, let us recall a result in [2], which unifies the existence and uniqueness results obtained in [3] and [5]. For this, let us introduce the following assumptions:

(H3') g has a general growth with respect to y , i.e, $dP \times dt - a.e.$,

$$\forall y \in \mathbb{R}^k, \quad |g(\omega, t, y, 0)| \leq |g(\omega, t, 0, 0)| + \varphi(|y|),$$

where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing continuous function.

(H5') $\mathbb{E} \left[\int_0^T |g(\omega, t, 0, 0)|^2 dt \right] < +\infty$.

Proposition 2 (see Theorem 2.1 in [2]) Let assumptions (H1), (H2), (H3'), (H4) and (H5') hold. Then for each $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^k)$, BSDE (ξ, T, g) has a unique solution.

The following Theorem 1 is the main result of this paper.

Theorem 1 Let assumptions (H1)-(H5) hold. Then for each $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^k)$, BSDE (ξ, T, g) has a unique solution.

Remark 2 Note that (H3) and (H5) are strictly weaker than (H3') and (H5') respectively. It is clear that Theorem 1 generalizes Proposition 2 and the corresponding results in [3, 5].

Example 1 Let $k = 2$ and for each $y = (y_1, y_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{2 \times d}$, let $g(t, y, z) = (g_1(t, y, z), g_2(t, y, z))$ be defined by

$$g_i(t, y, z) = |B_t| \cdot e^{-y_i} + h(|y|) + |z| + \frac{1}{\sqrt{t}} \cdot 1_{t>0}, \quad i = 1, 2,$$

where

$$h(x) = \begin{cases} -x \ln x & , \quad 0 < x \leq \delta; \\ h'(x)(x - \delta) + h(\delta) & , \quad x > \delta; \\ 0 & , \quad \text{other cases} \end{cases}$$

with $\delta > 0$ small enough.

It is not hard to check that this g satisfies (H3) and (H5), but does not satisfy (H3') and (H5'). At the same time, it is clear that g satisfies (H2) and (H4) with $\mu = 1$. In addition, we can also prove that g satisfies (H1) (see Examples 2.4-2.5 and Remark 2.2 in [2] for details). Then, it follows from Theorem 1 that for each $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^k)$, BSDE(ξ, T, g) has a unique solution. It should be mentioned that this conclusion can not be obtained by any known results including the previous proposition 2.

The Proof of Theorem 1. Assume that g satisfies assumptions (H1)-(H5). The proof of the uniqueness part is similar to that of the uniqueness part of Theorem 2.1 in [2], so we omit it. Let us turn to the existence part. The proof will be split into two steps.

First step: We shall prove that under assumptions (H1)-(H5), provided that there exists a constant $K > 0$ such that

$$dP - a.s., \quad |\xi| \leq K, \quad \text{and} \quad dP \times dt - a.e., \quad |g(t, 0, 0)| \leq K, \quad (7)$$

BSDE (ξ, T, g) has a solution.

For some large enough integer $\alpha > 0$ which will be chosen later, let θ_α be a smooth function such that $0 \leq \theta_\alpha \leq 1$, $\theta_\alpha(y) = 1$ for $|y| \leq \alpha$ and $\theta_\alpha(y) = 0$ as soon as $|y| \geq \alpha + 1$. For each $n \geq 1$ and $z \in \mathbb{R}^{k \times d}$, we denote $q_n(z) = zn/(|z| \vee n)$ and set

$$h_n(t, y, z) := \theta_\alpha(y)(g(t, y, q_n(z)) - g(t, 0, 0)) \frac{n}{\phi_{\alpha+1}(t) \vee n} + g(t, 0, 0),$$

where $\phi_\alpha(\cdot)$ is defined in (H3).

It is clear from (7) that h_n satisfies assumptions (H2) and (H4) and (H5') for each $n \geq 1$. It is also easy from (7) and (H3) to check that $|h_n(t, y, 0)| \leq n + K$, which means that h_n satisfies (H3'). We now prove that h_n satisfies also assumption (H1) but with another concave function $\bar{\kappa}(\cdot)$ which will be chosen later. Indeed, let us pick y_1 and y_2 in \mathbb{R}^k . If $|y_1| > \alpha + 1$ and $|y_2| > \alpha + 1$, (H1) is trivially satisfied and thus we reduce to the case where $|y_2| \leq \alpha + 1$.

We write

$$\begin{aligned}
& \langle y_1 - y_2, h_n(t, y_1, z) - h_n(t, y_2, z) \rangle \\
&= \theta_\alpha(y_1) \frac{n}{\phi_{\alpha+1}(t) \vee n} \langle y_1 - y_2, g(t, y_1, q_n(z)) - g(t, y_2, q_n(z)) \rangle \\
&\quad + \frac{n}{\phi_{\alpha+1}(t) \vee n} (\theta_\alpha(y_1) - \theta_\alpha(y_2)) \langle y_1 - y_2, g(t, y_2, q_n(z)) - g(t, 0, 0) \rangle.
\end{aligned}$$

Since g satisfies (H1), the first term of the right-hand side of the previous equality is smaller than the term $\kappa(|y_1 - y_2|^2)$. For the second term, we can use the fact that θ_α is $C(\alpha)$ -Lipschitz, to get, since $|y_2| \leq \alpha + 1$,

$$\begin{aligned}
& (\theta_\alpha(y_1) - \theta_\alpha(y_2)) \langle y_1 - y_2, g(t, y_2, q_n(z)) - g(t, 0, 0) \rangle \\
&\leq C(\alpha) |y_1 - y_2|^2 |g(t, y_2, q_n(z)) - g(t, 0, 0)| \leq C(\alpha) (\phi_{\alpha+1}(t) + \mu n) |y_1 - y_2|^2
\end{aligned}$$

and thus

$$\frac{n}{\phi_{\alpha+1}(t) \vee n} (\theta_\alpha(y_1) - \theta_\alpha(y_2)) \langle y_1 - y_2, g(t, y_2, q_n(z)) - g(t, 0, 0) \rangle \leq C(\alpha) (1 + \mu) n |y_1 - y_2|^2.$$

Hence, letting $\bar{\kappa}(x) = C(\alpha)(1 + \mu)nx + \kappa(x)$, we have

$$\langle y_1 - y_2, h_n(t, y_1, z) - h_n(t, y_2, z) \rangle \leq \bar{\kappa}(|y_1 - y_2|^2).$$

It is clear that $\bar{\kappa}(\cdot)$ is a nondecreasing concave function with $\bar{\kappa}(0) = 0$ and $\bar{\kappa}(u) > 0$ for $u > 0$. Moreover, it follows from the concavity of $\kappa(\cdot)$ that

$$\kappa(u) = \rho(u \cdot 1 + (1 - u) \cdot 0) \geq u\kappa(1) + (1 - u)\rho(0) = u\kappa(1), \quad u \in [0, 1],$$

and then

$$\int_{0+} \frac{du}{\bar{\kappa}(u)} = \int_{0+} \frac{du}{C(\alpha)(1 + \mu)nu + \kappa(u)} \geq \frac{\kappa(1)}{C(\alpha)(1 + \mu)n + \kappa(1)} \int_{0+} \frac{du}{\kappa(u)} = +\infty.$$

Then the pair (ξ, h_n) satisfies all the assumptions of Proposition 2. Hence, for each $n \geq 1$, BSDE (ξ, T, h_n) has a unique solution $(y_t^n, z_t^n)_{t \in [0, T]}$.

Furthermore, it follows from (H1), (H4) and (7) that

$$\begin{aligned}
\langle y, h_n(t, y, z) \rangle &= \theta_\alpha(y) \frac{n}{\phi_{\alpha+1}(t) \vee n} \langle y, g(t, y, q_n(z)) - g(t, 0, q_n(z)) \\
&\quad + g(t, 0, q_n(z)) - g(t, 0, 0) \rangle + \langle y, g(t, 0, 0) \rangle \\
&\leq \kappa(|y|^2) + \mu|y||z| + K|y|.
\end{aligned}$$

Consequently, assumption (A) is satisfied for the generator h_n of BSDE (ξ, T, h_n) with $\psi(u) = \kappa(u)$, $\lambda = \mu$ and $f_t \equiv K$. It then follows from Proposition 1 and (7) that there exists a constant $C > 0$ depending only on μ and T such that for each $0 \leq u \leq t \leq T$,

$$\mathbb{E} \left[\sup_{r \in [t, T]} |y_r^n|^2 \middle| \mathcal{F}_u \right] + \mathbb{E} \left[\int_t^T |z_s^n|^2 ds \middle| \mathcal{F}_u \right] \leq CK^2(1 + T^2) + C \int_t^T \kappa(\mathbb{E}[|y_s^n|^2 | \mathcal{F}_u]) ds.$$

Since $\kappa(\cdot)$ is a nondecreasing and concave function with $\kappa(0) = 0$, it increases at most linearly, i.e., there exists a constant $A > 0$ such that $\kappa(u) \leq A(u + 1)$ for each $u \geq 0$. Applying Gronwall's inequality to the previous inequality yields that

$$\mathbb{E} [|y_t^n|^2 | \mathcal{F}_u] + \mathbb{E} \left[\int_t^T |z_s^n|^2 ds \middle| \mathcal{F}_u \right] \leq \alpha^2,$$

where $\alpha := \sqrt{CK^2(1+T^2) + CAT} \cdot e^{CAT/2}$. Substituting $u = t$ in the previous inequality yields that for each $n \geq 1$ and $t \in [0, T]$,

$$|y_t^n| \leq \alpha, \quad \text{and} \quad \mathbb{E} \left[\int_0^T |z_s^n|^2 ds \right] \leq \alpha^2. \quad (8)$$

As a byproduct, $(y_t^n, z_t^n)_{t \in [0, T]}$ solves the BSDE (ξ, T, g_n) , where

$$g_n(t, y, z) = (g(t, y, q_n(z)) - g(t, 0, 0)) \frac{n}{\phi_{\alpha+1}(t) \vee n} + g(t, 0, 0).$$

In the sequel, for each $n \geq 1$ and $i \geq 1$, let $\hat{y}^{n,i} = y^{n+i} - y^n$, $\hat{z}^{n,i} = z^{n+i} - z^n$. We have

$$\hat{y}_t^{n,i} = \int_t^T \hat{g}^{n,i}(s, \hat{y}_s^{n,i}, \hat{z}_s^{n,i}) ds - \int_t^T \hat{z}_s^{n,i} dB_s, \quad t \in [0, T],$$

where for each $y \in \mathbb{R}^k$,

$$\begin{aligned} \hat{g}^{n,i}(s, y, z) &:= (g(s, y + y_s^n, q_{n+i}(z + z_s^n)) - g(s, 0, 0)) \frac{(n+i)}{\phi_{\alpha+1}(s) \vee (n+i)} \\ &\quad - (g(s, y_s^n, q_n(z_s^n)) - g(s, 0, 0)) \frac{n}{\phi_{\alpha+1}(s) \vee n}. \end{aligned}$$

It also follows from (H1) and (H4) that

$$\begin{aligned} \langle y, \hat{g}^{n,i}(s, y, z) \rangle &= \frac{(n+i)}{\phi_{\alpha+1}(s) \vee (n+i)} \langle y, g(s, y + y_s^n, q_{n+i}(z + z_s^n)) - g(s, y_s^n, q_n(z_s^n)) \rangle \\ &\quad + \left(\frac{(n+i)}{\phi_{\alpha+1}(s) \vee (n+i)} - \frac{n}{\phi_{\alpha+1}(s) \vee n} \right) \langle y, g(s, y_s^n, q_n(z_s^n)) - g(s, 0, 0) \rangle \\ &\leq \kappa(|y|^2) + \mu|y|(|z| + 2|z_s^n|1_{|z_s^n|>n}) + 21_{\phi_{\alpha+1}(s)>n}|y|(\phi_{\alpha+1}(s) + \mu|z_s^n|), \end{aligned} \quad (9)$$

where we have used the fact that

$$\begin{aligned} |q_{n+i}(z + z_s^n) - q_n(z_s^n)| &\leq |q_{n+i}(z + z_s^n) - q_{n+i}(z_s^n)| + |q_{n+i}(z_s^n) - q_n(z_s^n)| \\ &\leq |z| + 2|z_s^n|1_{|z_s^n|>n}. \end{aligned}$$

Then, combining (8), (9) and the inequality $2ab \leq 2a^2 + b^2/2$ we deduce that

$$\begin{aligned} 2\langle \hat{y}_s^{n,i}, \hat{g}^{n,i}(s, \hat{y}_s^{n,i}, \hat{z}_s^{n,i}) \rangle &\leq 2\kappa(|\hat{y}_s^{n,i}|^2) + 2\mu^2|\hat{y}_s^{n,i}|^2 + \frac{1}{2}|\hat{z}_s^{n,i}|^2 + 4\alpha\mu|z_s^n|1_{|z_s^n|>n} \\ &\quad + 4\alpha 1_{\phi_{\alpha+1}(s)>n}(\phi_{\alpha+1}(s) + \mu|z_s^n|). \end{aligned}$$

With this inequality in hand, using a similar argument to the proof of Proposition 3.1 in [2], we can deduce that there exists a constant $C > 0$ depending only on μ and T such that for each $t \in [0, T]$ and each $n, i \geq 1$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{r \in [t, T]} |\hat{y}_r^{n,i}|^2 + \int_t^T |\hat{z}_s^{n,i}|^2 ds \right] \\ & \leq C \int_t^T \kappa \left(\mathbb{E} \left[\sup_{r \in [s, T]} |\hat{y}_r^{n,i}|^2 \right] \right) ds + 2C\alpha\mu \mathbb{E} \left[\int_t^T |z_s^n| 1_{|z_s^n| > n} ds \right] \\ & \quad + 2C\alpha \mathbb{E} \left[\int_t^T 1_{\phi_{\alpha+1}(s) > n} (\phi_{\alpha+1}(s) + \mu|z_s^n|) ds \right]. \end{aligned}$$

Furthermore, with the help of (8), (H3) and the assumptions of $\kappa(\cdot)$, taking the limsup with respect to n in the previous inequality and using Fatou's lemma and Bihari's inequality yields that $\{(y_t^n, z_t^n)_{t \in [0, T]}\}_{n=1}^\infty$ is a Cauchy sequence in the process space $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$. Finally, we can pass to the limit in the approximating BSDE (ξ, T, g_n) , which yields a solution to BSDE (ξ, T, g) .

Second step: We now treat the general case. For each $n \geq 1$, let

$$\xi_n := q_n(\xi) \quad \text{and} \quad g_n(t, y, z) := g(t, y, z) - g(t, 0, 0) + q_n(g(t, 0, 0)). \quad (10)$$

Clearly, the (ξ_n, g_n) satisfies the assumptions of the first step and

$$\mathbb{E} [|\xi_n - \xi|^2] \rightarrow 0, \quad \mathbb{E} \left[\left(\int_0^T |q_n(g(s, 0, 0)) - g(s, 0, 0)| ds \right)^2 \right] \rightarrow 0 \quad (11)$$

as $n \rightarrow \infty$ by (H5). For each $n \geq 1$, thanks to the first step of this proof, let $(y_t^n, z_t^n)_{t \in [0, T]}$ denote the unique solution to BSDE (ξ_n, T, g_n) . For each $n \geq 1$ and $m \geq 1$, let $\hat{y}^{n,m} = y^n - y^m$, $\hat{z}^{n,m} = z^n - z^m$, we have

$$\hat{y}_t^{n,m} = \xi_n - \xi_m + \int_t^T \hat{g}^{n,m}(s, \hat{y}_s^{n,m}, \hat{z}_s^{n,m}) ds - \int_t^T \hat{z}_s^{n,m} dB_s, \quad t \in [0, T], \quad (12)$$

where for each $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$\hat{g}^{n,m}(s, y, z) := g_n(s, y + y_s^m, z + z_s^m) - g_m(s, y_s^m, z_s^m).$$

We write

$$\begin{aligned} \langle y, \hat{g}^{n,m}(t, y, z) \rangle &= \langle y, g_n(t, y + y_t^m, z + z_t^m) - g_m(t, y + y_t^m, z + z_t^m) \rangle \\ &\quad + \langle y, g_m(t, y + y_t^m, z + z_t^m) - g_m(t, y_t^m, z_t^m) \rangle. \end{aligned}$$

It follows from (10), (H1) and (H4) that for each $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $dP \times dt - a.e.$,

$$\begin{aligned} \langle y, \hat{g}^{n,m}(t, y, z) \rangle &= \langle y, q_n(g(t, 0, 0)) - q_m(g(t, 0, 0)) \rangle \\ &\quad + \langle y, g(t, y + y_t^m, z + z_t^m) - g(t, y_t^m, z_t^m) \rangle \\ &\leq |y| |q_n(g(t, 0, 0)) - q_m(g(t, 0, 0))| + \kappa(|y|^2) + \mu|y||z|. \end{aligned}$$

Consequently, assumption (A) is satisfied for the generator $\hat{g}^{n,m}(t, y, z)$ of BSDE (12) with $\psi(u) = \kappa(u)$, $\lambda = \mu$ and $f_t = |q_n(g(t, 0, 0)) - q_m(g(t, 0, 0))|$. It then follows from Proposition 1 with $u = 0$ that there exists a constant $C > 0$ depending only on T and μ such that for each $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{r \in [t, T]} |\hat{y}_r^{n,m}|^2 \right] + \mathbb{E} \left[\int_t^T |\hat{z}_s^{n,m}|^2 ds \right] \\ & \leq C \mathbb{E} [|\xi_n - \xi_m|^2] + C \int_t^T \kappa \left(\mathbb{E} \left[\sup_{r \in [s, T]} |\hat{y}_r^{n,m}|^2 \right] \right) ds \\ & \quad + C \mathbb{E} \left[\left(\int_0^T |q_n(g(s, 0, 0)) - q_m(g(s, 0, 0))| ds \right)^2 \right]. \end{aligned} \quad (13)$$

Note that there exists a constant $A > 0$ such that $\kappa(u) \leq A(u + 1)$ for each $u \geq 0$. Gronwall's inequality yields that for each $t \in [0, T]$ and each $n, m \geq 1$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{r \in [t, T]} |\hat{y}_r^{n,m}|^2 \right] + \mathbb{E} \left[\int_t^T |\hat{z}_s^{n,m}|^2 ds \right] \\ & \leq e^{CAT} \cdot \left(4C \mathbb{E} [|\xi|^2] + CAT + 4C \mathbb{E} \left[\left(\int_0^T |g(s, 0, 0)| ds \right)^2 \right] \right). \end{aligned}$$

Thus, in view of (11), by taking the limsup in (13) with respect to n, m and using Fatou's lemma, the monotonicity and continuity of $\kappa(\cdot)$ and Bihari's inequality we know that $\{(y_t^n, z_t^n)_{t \in [0, T]}\}_{n=1}^\infty$ is a Cauchy sequence in the process space $\mathcal{S}^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$. Let $(y_t, z_t)_{t \in [0, T]}$ be the limit process of the sequence $\{(y_t^n, z_t^n)_{t \in [0, T]}\}_{n=1}^\infty$. We pass to the limit in uniform convergence in probability for BSDEs (ξ_n, T, g_n) , thanks to (H2), (H3) and (H4), to see that $(y_t, z_t)_{t \in [0, T]}$ solves BSDE (ξ, T, g) . Thus, we complete the proof of Theorem 1.

References

- [1] Briand, Ph., Delyon, B., Hu, Y., Pardoux, E., Stoica, L., 2003. L^p solutions of backward stochastic differential equations. *Stochastic Processes and their Applications*, 108:109-129.
- [2] Fan, S., Jiang, L., 2013. Multidimensional BSDEs with weakly monotonic generators. *Acta Mathematica Sinica, English Series*, 29:1885-1906.
- [3] Mao, X., 1995. Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients. *Stochastic Process and their Applications*, 58:281-292.
- [4] Pardoux, E., Peng, S., 1990. Adapted solution of a backward stochastic differential equation. *Systems Control Letters*, 14:55-61.
- [5] Pardoux, E., 1999. BSDEs, weak convergence and homogenization of semilinear PDEs. *Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998)*. Kluwer Academic Publishers, Dordrecht, pp:503-549.